Probability

Stat 145

Probability Rules

1. **Probability Scale:** $0 \leq \mathrm{P}(A) \leq 1$ for any event A.
2. **Total Probability:** $\mathrm{P}(S) = 1$ where S is the sample space.
3. **Additivity:** If A and B are **mutually exclusive** events, then

$$\mathrm{P}(A \,\mathrm{or}\, B) = \mathrm{P}(A) + \mathrm{P}(B)\ .$$

   - Generalization: $\mathrm{P}(A_1 \,\mathrm{or}\, A_2 \,\mathrm{or} \cdots \mathrm{or}\, A_k) = \mathrm{P}(A_1) + \mathrm{P}(A_2) + \cdots + \mathrm{P}(A_k)$, provided all the events are mutuall exclusive.

4. **Equally Likely Rule**: If all outcomes are equally likely, then

$$\mathrm{P}(A) = \frac{\text{number of outcomes in A}}{\text{number of outcomes in sample space}}\ .$$

5. **Complement Rule**: $\mathrm{P}(A^c) = 1 - \mathrm{P}(A)$
6. **General Addition Rule**: $\mathrm{P}(A \,\mathrm{or}\, B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \,\mathrm{and}\, B)$
7. **Multiplication Rule**: $\mathrm{P}(A \,\mathrm{and}\, B) =$

Conditional Probability

Notation: $\mathrm{P}(A \,\mathrm{if}\, B)$ or $\mathrm{P}(A \mid B)$.

We will read this one of these ways:

- "the probability of A if B happens".
- "the probability of A if B".
- "the probability of A given that B happens".
- "the probability of A given B".

Asking the question: What proportion of the time that B happens does A also happen.

Definition:

$$\mathrm{P}(A \,\mathrm{if}\, B) = \frac{\mathrm{P}(A \,\mathrm{and}\, B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(\text{both})}{\mathrm{P}(\text{condition})}$$

- only defined if $\mathrm{P}(B) > 0$.

Exercises

1. If we roll a **fair die** (6 sides numbered 1 through 6, each equally likely to be rolled), what is

   a. the probability that we roll a 6?
   b. the probability that we do not roll a 6?
   c. the probability that we roll an even number?

2. **Cystic fibrosis** (CF) is a life-threatening genetic disorder caused by mutations in the CFTR gene located on chromosome 7. Defective copies of CFTR can result in the reduced quantity and function of the CFTR protein, which leads to the buildup of thick mucus in the lungs and pancreas. CF is an autosomal recessive disorder; an individual only develops CF if they have inherited two affected copies of CFTR (one from each parent). Individuals with one normal (wild-type) copy and one defective (mutated) copy are known as carriers; they do not develop CF, but may pass the disease-causing mutation onto their offspring.

   a. If both parents are carriers, what is the probability that their child will have CF?
   b. If both parents are carriers, what is the probability that their child will not have CF?
   c. If both parents are carriers, what is the probability that their child will be a carrier?
   d. If both parents are carriers, what is the probability that their child will either be a carrier or have CF?
   e. If one parent is a carrier and the other is not, what is the probability that their child will have CF?
   f. If one parent is a carrier and the other is not, what is the probability that their child will be a carrier?

A **standard deck of cards** has 52 cards. There are four suits (clubs, diamonds, hearts, spades) and 13 denominations in each suit (Ace, 2, 3, 4, 5, 6, 7, 8, 9, jack, queen, king). Clubs and spades are black. Diamonds are hearts are red.

3. If you are dealt 1 card from a standard deck, what is

   a. the probability that the card is a heart?
   b. the probability that the card is a jack?
   c. the probability that the card is a jack of hearts?
   d. the probability that the card is red?
   e. the probability that the card is a face card (jack, queen, or king)?

4. If we select one card from a shuffled deck of cards, what is the probability that the card

   a. is either a diamond or a heart?
   b. either a face card (jack, queen, or king) or and Ace?
   c. either a diamond or a face card?

5. **Family with Two Kids.** A family has two children. One of them is a boy. What is the probability that the other is a girl?

6. **T-shirts.** A survey of a 5th grade students at a school asked them whether they preferred Red or Yellow for a class T-shirt and also recorded whether each child was a boy or a girl. Let B be the event that a child is a boy, G that the child is a girl, R that the child prefers red, and Y that the child prefers yellow. Suppose we randomly select one survey respondent from the class. Compute each of these probabilities using the table below.

.	Red	Yellow
Boy	17	8
Girl	12	16

a. $\mathrm{P}(B)$

b. $\mathrm{P}(G)$

c. $\mathrm{P}(R)$

d. $\mathrm{P}(B \text{ and } R)$

e. $\mathrm{P}(R \text{ and } B)$

f. $\mathrm{P}(B \text{ or } R)$

g. $\mathrm{P}(R \text{ or } B)$

h. $\mathrm{P}(B \text{ if } R)$

i. $\mathrm{P}(R \text{ if } B)$

j. $\mathrm{P}(Y \text{ if } B)$

k. $\mathrm{P}(B \text{ if } Y)$

7. **The multiplication rule.**

a. By the definition of conditional probability what is $\mathrm{P}(B \text{ if } A)$?
b. Solve this equation for $\mathrm{P}(A \text{ and } B)$. Fill in the result above as Rule 7.
c. We can generalize this to more than two events. For example, use your rule on $\mathrm{P}(A \text{ and } B \text{ and } C)$ by thinking of it first as $\mathrm{P}((A \text{ and } B) \text{ and } C)$, then use it again on the $\mathrm{P}(A \text{ and } B)$ part of the result. What formula to you end up with for $\mathrm{P}(A \text{ and } B \text{ and } C)$?
d. What formula do we get for $\mathrm{P}(A \text{ and } B \text{ and } C \text{ and } D)$?
e. I won't make you do any more of these, but hopefully you see the pattern now. It works for any number of events.

8. **Independence.** Sometimes $\mathrm{P}(A \text{ if } B) = \mathrm{P}(A)$. That means knowing that B happens doesn't change our probability for A – it adds no information useful for computing the probability that A happens. When this happens, we say that A and B are **independent events**.[1]

   a. What is the **Multiplication Rule for Independent Events**?

   b. Toss a coin twice. Let H_1 be the event that the first toss is a head. Let H_2 be the even that the second toss is a head.

      - Are these independent events? Explain.
      - What is $\mathrm{P}(H_1 \text{ and } H_2)$?

   c. Shuffle a deck of cards and deal two of them. Let B_1 be the event that the first card is black. Let B_2 be the event that the second card is black.

      - Are these independent events? Explain.
      - What is $\mathrm{P}(B_1 \text{ and } B_2)$?

   d. Roll two standard dice. Let A be the event that the first die is even. Let B be the event that the second die is a six.

      - Are these independent events? Explain.
      - What is $\mathrm{P}(A \text{ and } B)$?

   e. Can mutually exclusive events be independent? Give an example or explain why not.

9. **Hospital.** Suppose that 35% of all patients admitted to a hospital's intensive care unit have high blood pressure, 42% have some sort of infection, and 12% have both problems.

   a. What is the probability that a person has at least one of these two conditions?
   b. What is the probability that the person has neither problem?
   c. What is that probability that a person with high blood pressure also has an infection?
   d. What is the probability that a person has high blood prssure if an infection is present?

10. If we draw two cards from a deck of cards, what is the probability that

    a. both are hearts b. both are aces c. neither is an ace d. one is a heart but not both

11. Flip a coin 4 times, what is the probability of getting heads each time? What if you flip it 10 times?

12. Suppose you conduct a hypothesis test with $\alpha = 0.05$ and the null hypothesis is true.

    a. What is the probability that your p-value will be less than α?
    b. What is the probability that your p-value will be greater than α?
    c. If you do ten such tests, each with different data, what is the probability all of the p-values will be greater than α?
    d. If you do ten such tests, each with different data, what is the probability that at least one of them will have a p-value less than α?

13. In a bowl are 4 red marbles and 2 blue marbles. If you reach in without looking and select two of the marbles, let X be the number of blue marbles. Fill in the following probability table.

value of X	0	1	2
probability			

    a. Do your probabilities add up to 1? Should they?

    b. Are the events $X = 0$, $X = 1$, and $X = 2$ equally likely?

[1] It always goes both ways. If $\mathrm{P}(A \text{ if } B) = \mathrm{P}(A)$, it will also be the case that $\mathrm{P}(B \text{ if } A) = \mathrm{P}(B)$. If you like to do a little algebra, you can show that this is so.

Some Answers and Solutions

6. T-shirts
   - $\mathrm{P}(B) = 25/53 = 0.472$
   - $\mathrm{P}(G) = 28/53 = 0.528$
   - $\mathrm{P}(R) = 29/53 = 0.547$
   - $\mathrm{P}(B \text{ and } R) = \mathrm{P}(R \text{ and } B) = 17/53 = 0.321$
   - $\mathrm{P}(B \text{ or } R) = \mathrm{P}(R \text{ or } B) = 37/53 = 0.698$
   - $\mathrm{P}(B \text{ if } R) = 17/29 = 0.586$
   - $\mathrm{P}(R \text{ if } B) = 17/25 = 0.68$
7. Multiplication Rule:
   - $\mathrm{P}(A \text{ and } B) = \mathrm{P}(A) \cdot \mathrm{P}(B \text{ if } A)$
   - $\mathrm{P}(A \text{ and } B \text{ and } C) = \mathrm{P}(A) \cdot \mathrm{P}(B \text{ if } A) \cdot \mathrm{P}(C \text{ if } A \text{ and } B$
8. If A and B are independent, then $\mathrm{P}(A \text{ and } B) = \mathrm{P}(A) \cdot \mathrm{P}(B)$. But this only holds if A and B are independent.
   - $\mathrm{P}(H_1 \text{ and } H_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ [independent]
   - $\mathrm{P}(B_1 \text{ and } B_2) = \frac{26}{52} \cdot \frac{25}{51} = 0.245$ [not independent]
   - $\mathrm{P}(A \text{ and } B) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$ [independent]
9. Hospital
   - at least one: $0.35 + 0.42 - 0.12 = 0.65$
   - neither: $1 - 0.65 = 0.35$
   - $\mathrm{P}(\text{infection} \mid \text{high blood pressure}) = 0.12/0.35 = 0.343$
   - $\mathrm{P}(\text{high blood pressure} \mid \text{infection}) = 0.12/0.42 = 0.286$
10. Cards
    - two hearts: 0.059
    - two aces: 0.005
    - no aces: 0.851
    - exactly one heart: 0.118
11. Coin
    - four heads: $(1/2)^4 = 0.062$
    - ten heads: $(1/2)^{10} = 9.766 \times 10^{-4}$
12. Hypothesis Test
    a. 0.05
    b. 0.95
    c. $0.95^{10} = 0.599$
    d. 0.401
13. Bowl of marbles
    - $\mathrm{P}(X = 0) = 0.4$
    - $\mathrm{P}(X = 1) = 0.533$
    - $\mathrm{P}(X = 2) = 0.067$
    - Not equally likely, but they do add up to 1.