5 Paths in Graphs

5.1 Definition of a path

- Informally, a **path** in a graph is a sequence of edges, each one incident to the next.
  - Can sometimes be described as a sequence of vertices, each one adjacent to the next (see below).
  - For directed graphs, we require that the directions of the edges be compatible.
- More formally, let n be a nonnegative integer and G an undirected [directed] graph. A **path** of **length** n from vertex v_0 to vertex v_n in G is a sequence of n edges $e_1, \ldots, e_n$ of G such that when $1 \leq i \leq n$, then $e_i = \{v_{i-1}, v_i\}$ $[e_i = \langle v_{i-1}, v_i \rangle]$
- A path is a **circuit** (also called a **cycle**) if it begins and ends at the same vertex and has length ≥ 1.
- A path or circuit is **simple** if it does not include the same edge more than once.

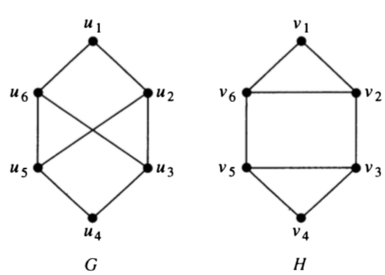
Questions

1. What is a path of length 0?
2. Can a path of length 1 be a circuit? If so, draw an example. If not, explain why not.
3. Why are paths defined in terms of edges rather than vertices? In what situations does it matter? When does it not matter?

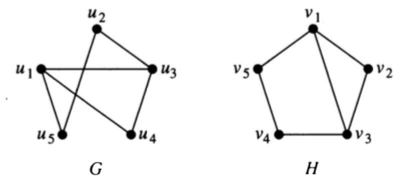
5.2 Paths and Isomorphism

Paths can be useful for finding an isomorphism between G and H or for showing that there is no isomorphism.

4. Consider the pair of graphs below.
   a. Compute the degreee sequences for each graph.
   b. Explain why any isophormphism must either pair u_1 with v_1 and u_4 with v_4 or else pair u_1 with v_4 and u_4 with v_1.
   c. How long is the shortest path from u_1 to u_4? the shortest path from v_1 to v_4?
   d. Is there a path of length 4 between u_1 and u_4? between v_1 and v_4?
   e. Are the graphs isomorphic? Explain.



5. How about these graphs? Are they isomporphic? How can considering paths help you figure this out?



5.3 Euler and Hamilton Paths

- An **Euler path** is a path that visits every edge of a graph exactly once.
- A **Hamilton path** is a path that visits every vertex exactly once.

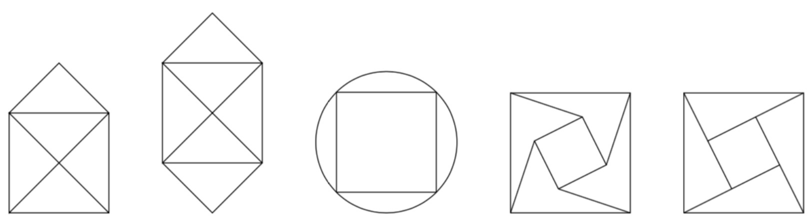
Euler paths are named after Leonid Euler who posed a famous problem about the bridges in Königsberg. Euler is pronounced "Oiler".

Hamilton paths are named after the Irish mathematician William Rowan Hamilton, not after the American politician Alexander Hamilton.

5.3.1 Can you trace it?

Perhaps you have seen the following type of "puzzle". The goal is to trace the figures without lifting your pencil from the paper and without retracing any of the lines.

6. Which kind of path is this puzzle asking for?
7. Which of these can you trace in this manner?



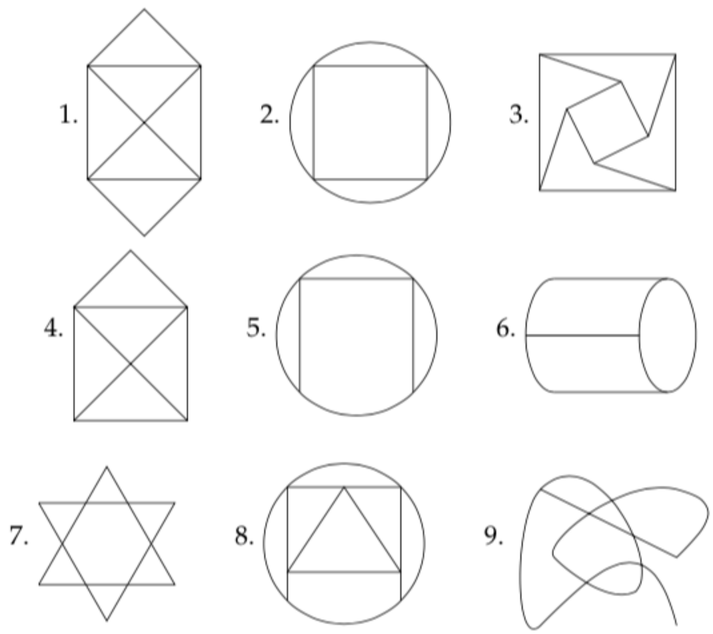
5.4 Relationships in Graphs

A **planar graph** is a graph that *can be* drawn in the plane without any edges crossing.

8. Show that K_4 is a planar graph by drawing it without any edge crossings.

9. Show that Q_3 is a planar graphs by drawing it without any edge crossings.

10. Here are some graphs that are missing their vertices. Add dots showing vertices so that each graph is a planar graph.



11. Fill in the table below for the graphs above. Do you notice any patterns? Can you make any conjectures? Can you prove any of the conjectures? (Note: an even vertex is a vertex with even degree. An odd vertex is a vertex with odd degree.)

Graph	number of even verticies	number of odd vertices	total degree	number of vertices	number of edges	number of regions	Euler path?	Euler circuit?
1								
2								
3								
4								
5								
6								
7								
8								
9								

5.5 Connectedness

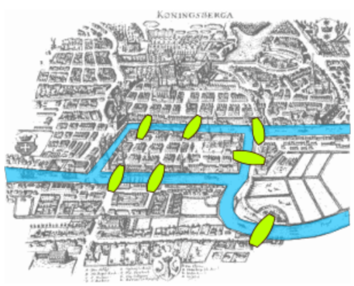
- An **undirected graph** is **connected** if there is a path between every pair of vertices.
- A **directed graph** is **strongly connected** if there is a path between every pair of vertices.
- A **directed graph** is **weakly connected** if the underlying undirected graph is connected.
- A connected component H of a graph G is maximal connected subgraph. That is,
  - H must be a subgraph of G,
  - H must be connected, but
  - H cannot be a subgraph of a larger connected subgraph of G.

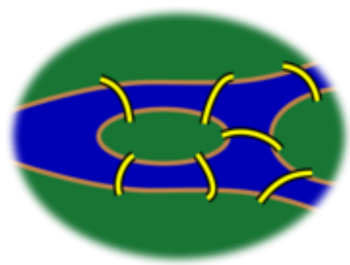
12. Draw a directed graph with 6 vertices that is weakly connected but not strongly connected.
13. Draw an undirected graph with two connected components. Count the number of vertices, edges, and regions in your graph. How does this compare to the results in the table in problem 11?

5.6 Königsberg Bridge Problem

5.6.1 Original Version

One of the founders of graph theory was Leonid Euler (one of the greatest mathematicians who ever lived). At the time he was alive, the city of Königsberg had seven bridges connecting the two banks of the river flowing through downtown and two islands in that river. Here are a couple maps of downtown Königsberg at the time. One is a bit stylized.





As the story goes, the residents of Königsberg enjoyed walking the bridges. The question is this: How can someone take a Sunday afternoon walk that starts and ends in the same place and crosses each bridge exactly once? (Leaving the map, renting hot air balloons, swimming, etc. are not allowed.)

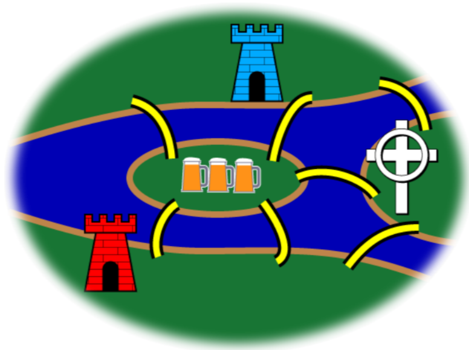
We'll return to this problem in just a moment. But first, a different sort of puzzle.

1. Create a graph that represents the Königsberg Bridge Puzzle (so it can be traced in this manner, if and only if the Königsberg Bridge Puzzle has a solution). What are the vertices? What are the edges? What type of graph is it?

2. Can you trace the Königsberg Bridge graph?

5.6.2 Extentions to the Königsberg bridge problem

Here are some extensions to the Königsberg Bridge problem. First, we embelish the map a bit: The northern bank of the river is occupied by the Schloss, or castle, of the Blue Prince; the southern by that of the Red Prince. The east bank is home to the Bishop's Kirche, or church; and on the small island in the center is a Gasthaus, or inn.



It being customary among the townsmen, after some hours in the Gasthaus, to attempt to walk the bridges, many have returned for more refreshment claiming to have walked each bridge exactly once. However, none have been able to repeat the feat by the light of day.

3. **The Blue Prince's eighth bridge.** The Blue Prince, having analyzed the town's bridge system by means of graph theory, concludes that the bridges cannot be walked. He contrives a stealthy plan to build an eighth bridge so that he can begin in the evening at his castle, walk the bridges, and end at the Gasthaus to brag of his victory. Of course, he wants the Red Prince to be unable to duplicate the feat. Where does the Blue Prince build the eighth bridge?

4. **The Red Prince's ninth bridge.** The Red Prince, infuriated by his brother's solution to the problem, wants to build a ninth bridge, enabling him to begin at his castle, walk the bridges, and end at the Gasthaus to rub dirt in his brother's face. Furthermore, his brother should then no longer walk the bridges himself. Where does the Red Prince build the ninth bridge?

5. **The Bishop's tength bridge.** The Bishop has watched this furious bridge-building with dismay. It upsets the townspeople and, worse, contributes to excessive drunkenness. He wants to build a tenth bridge that allows all the inhabitants to walk the bridges and return to their own beds. Where does the Bishop build the tenth bridge?