5 Probability

5.1 Notation Note

The notation used to probability varies a bit from book to book. Three common notations for the probability of an event E are $p(E)$, $\mathrm{P}(E)$ and $\Pr(E)$. Our book uses the lower case $p()$.

5.2 Probability Axioms

These three statements are the foundation of probability. Anything that follows these rules is a probability.

1. For any event E, $0 \leq p(E) \leq 1$.
2. If S is the sample space, then $p(S) = 1$.
3. The probability of a disjoint union is the sum of the probabilities.
   a. $p(A \cup B) = p(A) + p(B)$, if $A \cap B = \emptyset$.
   b. $p(A_1 \cup A_2 \cup \cdots \cup A_n) = p(A_1) + p(A_2) + \cdots + p(A_n)$, if $A_i \cap A_j = \emptyset$ whenever $i \neq j$.
   c. $p(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{n} p(A_i)$, if $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

5.3 Some Additional Probability Rules

These probability rules also hold for all probabilities.

4. **Complement Rule:** For any event E, $p(\overline{E}) = 1 - p(E)$
5. **Union Rule:** For any events A and B, $p(A \cup B) = p(A) + p(B) - p(A \cap B)$.
6. **Equally Likely Rule:** If all outcomes in a sample space are equally likely, then

$$p(E) = \frac{|E|}{|S|}$$

5.4 Exercises

1. Show how the Complement Rule follows from the axioms.
2. Show how the Union Rule follows from the axioms.
3. Show how the Equally Likely Rule follows from the axioms.

As you do the problems below, explicitly mention each of the 6 probability axioms/rules as you use them. (If you use the Equally Likely Rule, be sure to check that the outcomes are indeed equally likely.) For now, don't use any other rules (if you happen to know more) unless you first prove them using the rules above.

4. **Urn.** An urn contains 3 red balls and 5 blue balls. If you randomly select two balls, what is the probability that both are the same color?
5. **Dice.** If you role two standard dice, what is the probability that the sum of the dice is 5?
6. **Smaller Dice.** If you role two 4-sided dice, what is the probability that the sum of the dice is 5?
7. **6 or doubles.** If you roll two standard dice, what is the probability that you get either doubles or a sum equal to 6?
8. **Lotto.** Many lotteries offer a game that works like this: The player selects three digits (0–9). Later the lottery selects three digits. If the player's three digits match the lottery-selected three digits in order, the player wins a large prize. If the digits match, but not in the correct order, the player wins a smaller prize.
   a. If you pick the digits 1-2-3, what is your probability of winning the large prize? The small prize?

b. If you pick the digits 6-1-6, what is your probability of winning the large prize? The small prize?

c. What if you pick 7-7-7 (because 7 is your lucky number)?

9. **Flush.** If you are dealt 5 cards from a standard deck, what is the probability that they all have the same suit (clubs, diamonds, hearts, or spades)?

10. **4-of-a-kind.** If you are dealt 5 cards from a standard deck, what is the probability that four of them are the same kind? (By kind we mean a number 2–10, Jack, Queen, King, or Ace.)

11. **Random Bit String.** A bitstring of length 8 is selected at random (each bit string is equally likely). What is the probability that the string has at least one 0?

12. **Yahtzee.** In Yahtzee, five standard dice are rolled. If you roll 5 standard dice, what is the probability that

    a. all 5 numbers match? (That's called a Yahtzee).

    b. at least two numbers match? (In the game you get to re-roll some of the numbers, so if you have at least two that match, you have something to start from in an attempt to get a Yahtzee in subsequent rolls.)

13. **Loaded Dice.** A six-sided die has been "loaded" so that 6 is twice as likely to be rolled as any other number.

    a. What is the probability of rolling a 6 with such a die?
    b. What is the probability of rolling doubles with two such dice? (Doubles means two matching numbers.)
    c. What is the probability of rolling doubles with two standard dice?

14. **Boys and Girls.** A family has two children. One of them is a boy. What is the probability that the other is a girl?

15. **More Lottery.** In a lottery game, players select 6 distinct numbers from the numbers 1 – 40. (The order in which they are chosen doesn't matter – often you select the numbers by punching out tabs for the 6 numbers you want and there is no record of the order in which you punched them out.) Later, the lottery commission picks 6 numbers. Compute the probability of each of these events and put the results in a nice table.

    a. all 6 numbers match.
    b. exactly 5 of the 6 match.
    c. exactly 4 of the 6 match.
    d. exactly 3 of the 6 match.
    e. exactly 2 of the 6 match.
    f. exactly 1 of the 6 match.
    g. none of the 6 match.