10 More Expected Value

10.1 Transformations and Sums

10.1.1 Transformations

If X is a random variable and f is a function, then $f(X)$ is a new random variable such that

$$p(f(X) = b) = \sum_{\{a|f(a)=b\}} p(X = a)$$

This is actually much simpler than it looks. Basically, we just transform each possible value k that X can take into $f(k)$ and then collapse any cells in the probability table that end up having the same value.

Example. Below is a probability table for X. Let's create probability table for X^2.

value	1	2	3	4
probability	0.1	0.2	0.3	0.4

To do that, we simply square each of the values:

value	1	4	9	16
probability	0.1	0.2	0.3	0.4

The probabilities remain unchanged. The probability that $X = 3$ is 0.3, and in that case, $X^2 = 9$, so the probability that $X^2 = 9$ is also 0.3. (Note: It is important here that there are no other ways to get $X^2 = 9$ for this random variable. If X could be -3, we would add in that probability as well.)

OK. Your turn.

1. Using the probability table for X in the example, create a probability table for $5 - X$.
2. Below is a probability table for X. Create probability table for X^2 and $2X + 3$.

value	-1	0	1
probability	0.2	0.5	0.3

3. Below is a probability table for X. Create probability tables for X^2 and for $2X$.

value	0	1
probability	0.7	0.3

10.1.2 Sums

Consider a process that produces multiple numerical outcomes. (Example: roll a red die and a blue die. The number on the red die is one numerical result, the number on the blue die is a second numerical result.) We can represent[1]

$$p(X = a \wedge Y = b)$$

with a function of a and b or with a two-way table like this one:

	X = 1	X = 2	X = 3
Y = 1	0.10	0.10	0.25
Y = 2	0.10	0.05	0.10
Y = 3	0.10	0.15	0.05

Use the table above to answer the following questions.

4. What is $p(X = 3 \wedge Y = 2)$? What is $p(X = 3)$? What is $p(Y = 2)$?
5. What is $p(X = Y)$? What is $p(X > Y)$?
6. Create three probability tables, one for X, one for Y, and one for $X + Y$. [Hint: Start by determining the possible values.]
7. Now compute $\mathrm{E}(X)$, $\mathrm{E}(Y)$, and $\mathrm{E}(X + Y)$.

[1] $\wedge$ is mathematical notation for "and".

10.2 Linearity of Expected Value

Linearity Theorem for Expected Value. Expected value is linear in the following senses:

a. $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$

b. $\mathrm{E}(X_1 + X_2 + X_3 + \cdots + X_n) = \mathrm{E}(X_1) + \mathrm{E}(X_2) + \mathrm{E}(X_3) + \cdots + \mathrm{E}(X_n)$

10.2.1 Exercises

8. Let's make sure we understand what the linearity theorem says before moving on.
   a. If $\mathrm{E}(X) = 20$, what are $\mathrm{E}(3X + 2)$ and $\mathrm{E}(2X - 3)$?
   b. If $\mathrm{E}(X) = 5$ and $\mathrm{E}(Y) = 2$, what are $\mathrm{E}(X + Y)$ and $\mathrm{E}(X - Y)$?
   c. Are your results for #7 consistent with the theorem? Explain.

9. Let S be the sum on two fair 4-sided dice. Compute $\mathrm{E}(S)$ two ways: (a) by creating a probability table for S, and (b) by writing $S = X + Y$ and using the Linearity Theorem.

10. What is the expected value of the sum of two fair 6-sided dice? (You only have to do this one one way.)

11. Here is a more interesting example. Suppose 5 people put their names into a hat. Then each person pull one name out. Let X be the number of people who draw their own name. What is $\mathrm{E}(X)$? How would the answer change if 10 people put their names into the hat? 20? 1? [Hint: How can you write X as a sum?]

12. Let $X \sim \mathsf{Binom}(1, p)$. What is $\mathrm{E}(X)$? What is $\mathrm{E}(X^2)$? [Hint: Make a probability table. What are the possible values of X?]

13. Let $X \sim \mathsf{Binom}(n, p)$. What is $\mathrm{E}(X)$? [Hint: Use the previous problem and the Linearity Theorem. How can you write X as a sum?]

14. What is the expected number of diamonds in a 5-card hand dealt from a standard deck?

15. If you roll 5 4-sided dice, what is the expected number of 4s rolled?

16. If you roll 5 standard dice, what is the expected number of unique numbers rolled?

17. Can you prove the linearity theorem? [For part b, you may restrict your attention to a sum of two random variables.]