Counting and Probability

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Preface

These exercises were assembled to accompany a Discrete Mathematics course at Calvin College.

1 Some Counting Problems

Each of these problems involves determining how many of something there are. For most, there will be too many to simply make a list and count them off, so you will need to "count without counting". As you go along, make a list of important counting principles you are using.

1. A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

2. The chairs of an auditorium are labeled with a letter (indicating the row) and a positive integer not exceeding 100 (indicating the seat within the row). Give an upper bound on the number of seats in the auditorium.

3. There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

4. How many bitstrings of length 8 are there?

5. How many bitstrings of length 8 either start with a 1 or end with a 1?

6. A state issues license plates that consist of three letters followed by three numbers. How many such plates are there?

7. A graph isomorphism is a 1-1 and onto function from the vertices in one graph to the vertices in another (that additionally preserves edges). You decide to write a little program that will check each 1-1 and onto function to see if it an isomorphism.

   If your graphs have n vertices, how many such functions must you check?

8. Now you modify your graph isomorphism algorithm so that it only tries bijections (1-1 and onto functions) that respect the degrees of the vertices. If the degree sequences of two graphs are both $(5, 5, 4, 4, 4, 3, 3, 2)$, how many functions must you check? How much better is this than your previous algorithm (on these two graphs)?

9. Let's use the following coding scheme to describe possible telephone numbers.

   - let X denote a digit that can take on any integer value $0 - 9$;
   - let N denote a digit that can take on values $2 - 9$
   - let Y denote a digit that can only take on values 0 or 1.

   a. In the scheme used in the 1960's, telephone numbers had to have the form NYX-NNX-XXXX. How many such phone numbers are there?

   b. A new scheme (*The North American Numbering Plan*) was introduced that required telephone numbers have the form NXX-NXX-XXXX. How many more phone numbers are available in the new scheme?

10. How many password of length 6, 7, or 8 are there if the passwords must be alpha-numeric, capitalization doesn't matter, and there must be at least one digit?

11. In the previous example, how many passwords are there if the first character must be a letter?

12. Professor Pruim puts the names of 30 students into a hat. He draws one out and gives that person a chocolate bar. Then he draws out another name, and gives that person a bag of M and M's. Finally, he draws out a third name and gives that person a bag of Valentine hearts he picked up cheap on February 15.

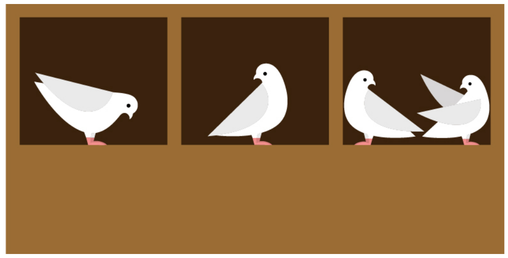
    a. How many different ways can the prizes be distributed?

    b. If you are one of the 30 students, how many of those ways include you getting a prize?

Answers/Solutions

1. `12 * 11` = 132.
2. `26 * 100 = 2600`
3. `32 * 24` = 768.
4. $2^8 = 256$
5. Two ways:
   a. only bad thing is beginning AND ending with 0, so $2^8 - 2^6 = 192$.
   b. Start with 1 or end with 1 but not both: $2^7 + 2^7 - 2^6 = 192$
6. $10^3 \cdot 26^3 = 17576000$
7. $n!$
8. $2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 1 = 24$. ($8! = 40320$, so that's a pretty big speed up.)
9. See book.
10. $36^6 - 26^6 + 36^7 - 26^7 + 36^8 - 26^8 = 2684483063360$
11. `26 * (36^5 - 26^5) + 26 * (36^6 - 26^6) + 26 * (36^7 - 26^7)` = 1878468937280
12. ```
    a. `30 * 29 * 28` = 24360
    b. `1 * 29 * 28 + 29 * 1 * 28 + 29 * 28 * 1` = `30 * 29 * 28 - 29 * 28 * 27`
    2436
    ```
    ```

2 Pigeonhole Principle



Big Idea: It is not possible for everyone to be above average or for everyone to be below average. (It is possible for everyone to be exactly average.)

So if n things are distributed into k groups,

- The largest group will have at least $\lceil n/k \rceil$ things, and
- The smallest group will have at most $\lfloor n/k \rfloor$ things.

This is called the (generalized) pigeon hole principle based on the analogy of putting letters into mail slots (called pigeon holes).

Applying the pigeonhole principle is generally easy *once you figure out what the pigeons (letters, really) and the pigeonholes are*. Figuring out the pigeons and holes can be easy, or it can take a clever idea. (Some famous proofs center on very clever uses of the pigeonhole principle.)

1. Suppose (in some alternative reality) that Professor Pruim brought 150 cookies to a class with 34 students. At the end of the hour, all of the cookies had been eaten by the students.

   a. What is the average number of cookies eaten by a student?
   b. This means the at least one student had at least ___________ cookies.
   c. This means the at least one student had at most ___________ cookies.

In each of the following problems, each time you use the pigeonhole principle, identify the "pigeons" and the "holes" and explain how that solves the problem. If you don't use the pigeonhole principle directly, be sure to explain your reasoning carefully.

2. Fill in the blanks.

   a. In a class of 38 students, there are always at least ________ students who were born in the same month.

   b. In a class of 38 students, there are always at least ________ students who were born on the same day of the week.

3. In the 2014 Winter Olympics, 88 nations competed in 98 events. Explain how this guaranteed that some country would win more than one gold medal. (No, the answer is not "Norway was there.")

4. At the 2014 Winter Olympics, Norway won 11 gold medals. Does that fact alone guarantee that a country won no gold medals?

5. Joe has 21 white socks and 15 black socks. (Don't ask what happened to the missing socks, no one knows where they go.)
   a. If he pulls socks in the dark (so he can't tell what color they are), how many must he grab to make sure he has a pair of white socks?
   b. If he pulls socks in the dark (so he can't tell what color they are), how many must he grab to make sure he has a pair of black socks?
   c. If he pulls socks in the dark (so he can't tell what color they are), how many must he grab to make sure he has a pair of matching socks?
6. For his birthday, Joe received a gift of 8 red socks (4 pairs, and he hasn't lost any yet). Now how many socks must Joe grab to be sure that he has a matching pair?
7. Joe is going on a trip and needs 3 pairs of matching socks. He's still too lazy to turn on the light to see what socks he is grabbing. How many socks must Joe bring along to make sure he has 3 matching pairs? (He doesn't care how many pairs are red, white, or black.)
8. Checkers.
   a. Consider a 3×9 grid (like an odd-shaped checker board). Show that no matter how you place red and black checkers on the 27 squares, there will always be a rectangle whose 4 corners have the same color checker. (The rectangles must be at least 2×2; 1×1 rectangles don't have four different checkers on their corners.)
   b. Consider a 3×7 grid (like an odd-shaped checker board). Show that no matter how you place red and black checkers on the 21 squares, there will always be a rectangle whose 4 corners have the same color checker. (The rectangles must be at least 2×2; 1×1 rectangles don't have four different checkers on their corners.)
   c. Show how to place red and black checkers on a 3×6 grid without having any rectangles with mono-chromatic corners.
9. Show that there is an integer that is divisible by 37 and consists only of 1's and 0's in its decimal representation.

   Hint: Consider the numbers 1, 11, 111, 1111, 11111, etc. Show that two of them are congruent mod 37. How does that help you answer the question?
10. Was there anything special about the number 37 in the previous example? For what other divisors does this work? Explain.
11. Is there anything special about the digits 1 and 0 in the previous example? What other combinations of digits would work?
12. A computer lab has 15 workstations and 10 servers. Workstations can only access servers via direct connections, and each server can only use one of these connections at a time.
    a. If we connect every workstation to every server, how many connections is that?
    b. Show that 60 connections suffice make it possible to use any set of 10 workstations simultaneously. (No matter what we do, we won't be able to connect 11 workstations to servers since there are only 10 servers.)
    c. Show that 59 connections is not enough.
       That is, no matter how you do the connecting, there will always be a combination of 10 workstations that cannot simultaneously connect to servers. (Hint: on average, how many connections are there per server?)
13. The integers from 1 to 10 are distributed around a circle. Prove that there must be three neighbors whose sum is at least 17. What about 18? What about 19?

3 Division Principle

3.1 A problem about cookies

1. Suppose 100 cookies are distributed equally to a group of kids. Each kid receives 4 cookies. How many kids are there?

3.2 The Division Principle

The reasoning used to solve the previous problem is called the division principle: If you count each item the same number of times, you can divide the count by the number of times each item is counted to get the number of items.

We have actually seen this before:

2. If the total degree of a graph is 58, how many edges does it have?

When we count the degrees, we count each edge twice, so there are $\frac{58}{2} = 29$ edges in the graph.

3.3 More Counting Problems

Here are some additional problems that involve our counting principles, including the division principle.

3. An NBA basketball team has 12 players.
   a. How many ways can the coach select the five players to start the game? (Positions don't matter.)
   b. How many ways can the coach select one center, two forwards, and two guards? (That is, the positions matter.)
4. How many 5-card poker hands can be dealt from a standard 52-card deck?
5. A class of 36 gets into groups of 4 to work on some problems.
   a. How many such groups of 4 are there?
   b. You get to work with 3 other people. How many such groups of 3 are there?
6. How many bit strings of length 8 have exactly three 1's?
7. How many "words" can you make with the letters in MISSISSIPPI? (The "words" do not need to be real words.)

4 Counting Problems

4.1 Donut problems

The problems below all involved donuts and that numbers 3 and 12. But you will need to know more than that to answer them.

1. Danielle's Donuts sells 12 kinds of donuts. This week they are offering a special: The Mighty Mixer. You get three donuts of three different kinds for $1.99. How many different Might Mixer orders are there?
2. Debra has purchased a Mighty Mixer consisting of one sprinkle donut, one jelly-filled donut, and one pumpkin spice donut. She decides to give them away to some of her friends. She has twelve friends.
   a. How many different ways can she give away the donuts if no one gets more than one?
   b. How many different ways can she give away the donuts if she allow the possibility of giving more than one to the same person?
3. Don's Donuts sells 3 kinds of donuts: plain, chocolate covered, and glazed. You decide to order a different dozen (12) every week until you have ordered every possible dozen. How many weeks will that take?
4. What are the key features that make these situations different from each other? Can you work out a general formula for each of the situations?
5. Write an additional problem of each type (and work out the answer). Do not use the numbers 3 or 12 in your problems. Do not use donuts.

4.2 More Problems

As you do these problems, identify where you are using our counting principles: Multiplication Principle, Addition Principle, Subtraction Principle, Division Principle, Inclusion-Exclusion Principle, Pigeonhole Principle. For problems in the "Donut Family", identify which of the four situations the problem represents.

6. How many bit strings of length eight either start with a 1 bit or end with the two bits 00?
7. How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?
8. A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?
9. These questions involve a standard deck of cards (52 cards, 13 each are hearts, diamonds, clubs, and spades).
   a. How many cards must be selected to guarantee that at least three of the cards are the same suit?
   b. How many must be selected to guarantee that at least three hearts are selected?
10. What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? How many additional phones can be accommodated before needing to add another area code? (Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)
11. Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

12. Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?
13. How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

5 Probability

5.1 Notation Note

The notation used to probability varies a bit from book to book. Three common notations for the probability of an event E are $p(E)$, $\mathrm{P}(E)$ and $\Pr(E)$. Our book uses the lower case $p()$.

5.2 Probability Axioms

These three statements are the foundation of probability. Anything that follows these rules is a probability.

1. For any event E, $0 \le p(E) \le 1$.
2. If S is the sample space, then $p(S) = 1$.
3. The probability of a disjoint union is the sum of the probabilities.
   a. $p(A \cup B) = p(A) + p(B)$, if $A \cap B = \emptyset$.
   b. $p(A_1 \cup A_2 \cup \cdots \cup A_n) = p(A_1) + p(A_2) + \cdots + p(A_n)$, if $A_i \cap A_j = \emptyset$ whenever $i \neq j$.
   c. $p(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{n} p(A_i)$, if $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

5.3 Some Additional Probability Rules

These probability rules also hold for all probabilities.

4. **Complement Rule:** For any event E, $p(\overline{E}) = 1 - p(E)$
5. **Union Rule:** For any events A and B, $p(A \cup B) = p(A) + p(B) - p(A \cap B)$.
6. **Equally Likely Rule:** If all outcomes in a sample space are equally likely, then

$$p(E) = \frac{|E|}{|S|}$$

5.4 Exercises

1. Show how the Complement Rule follows from the axioms.
2. Show how the Union Rule follows from the axioms.
3. Show how the Equally Likely Rule follows from the axioms.

As you do the problems below, explicitly mention each of the 6 probability axioms/rules as you use them. (If you use the Equally Likely Rule, be sure to check that the outcomes are indeed equally likely.) For now, don't use any other rules (if you happen to know more) unless you first prove them using the rules above.

4. **Urn.** An urn contains 3 red balls and 5 blue balls. If you randomly select two balls, what is the probability that both are the same color?
5. **Dice.** If you role two standard dice, what is the probability that the sum of the dice is 5?
6. **Smaller Dice.** If you role two 4-sided dice, what is the probability that the sum of the dice is 5?
7. **6 or doubles.** If you roll two standard dice, what is the probability that you get either doubles or a sum equal to 6?

8. **Lotto.** Many lotteries offer a game that works like this: The player selects three digits (0–9). Later the lottery selects three digits. If the player's three digits match the lottery-selected three digits in order, the player wins a large prize. If the digits match, but not in the correct order, the player wins a smaller prize.

   a. If you pick the digits 1-2-3, what is your probability of winning the large prize? The small prize?

   b. If you pick the digits 6-1-6, what is your probability of winning the large prize? The small prize?

   c. What if you pick 7-7-7 (because 7 is your lucky number)?

9. **Flush.** If you are dealt 5 cards from a standard deck, what is the probability that they all have the same suit (clubs, diamonds, hearts, or spades)?

10. **4-of-a-kind.** If you are dealt 5 cards from a standard deck, what is the probability that four of them are the same kind? (By kind we mean a number 2–10, Jack, Queen, King, or Ace.)

11. **Random Bit String.** A bitstring of length 8 is selected at random (each bit string is equally likely). What is the probability that the string has at least one 0?

12. **Yahtzee.** In Yahtzee, five standard dice are rolled. If you roll 5 standard dice, what is the probability that

    a. all 5 numbers match? (That's called a Yahtzee).

    b. at least two numbers match? (In the game you get to re-roll some of the numbers, so if you have at least two that match, you have something to start from in an attempt to get a Yahtzee in subsequent rolls.)

13. **Loaded Dice.** A six-sided die has been "loaded" so that 6 is twice as likely to be rolled as any other number.

    a. What is the probability of rolling a 6 with such a die?
    b. What is the probability of rolling doubles with two such dice? (Doubles means two matching numbers.)
    c. What is the probability of rolling doubles with two standard dice?

14. **Boys and Girls.** A family has two children. One of them is a boy. What is the probability that the other is a girl?

15. **More Lottery.** In a lottery game, players select 6 distinct numbers from the numbers 1 – 40. (The order in which they are chosen doesn't matter – often you select the numbers by punching out tabs for the 6 numbers you want and there is no record of the order in which you punched them out.) Later, the lottery commission picks 6 numbers. Compute the probability of each of these events and put the results in a nice table.

    a. all 6 numbers match.
    b. exactly 5 of the 6 match.
    c. exactly 4 of the 6 match.
    d. exactly 3 of the 6 match.
    e. exactly 2 of the 6 match.
    f. exactly 1 of the 6 match.
    g. none of the 6 match.

6 Conditional Probability

$p(A \mid B)$ answers the question: Of the times that B happens, how often does A also happen? Common ways this is expressed include

- The probability of A given B
- The probability of A conditional on B
- The probability of A if B
- The probability that A happens when B happens

When $p(B) \neq 0$, then

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)}$$

Note: Usually, $p(A \mid B)$, $p(B \mid A)$ and $p(A \cap B)$ are all different. It is critical to know which of these three applies in a given situation and to use the notation correctly.

6.1 Exercises

A number of school children, some boys and some girls, are asked about their favorite color. The resuls are in the table below.

\	Blue	Other
Boy	18	7
Girl	12	8

Suppose we put the names of all the children into a hat and selection one randomly. Consider the following events.

- G: The selected child is a girl.
- L: The selected child's favorite color is blue.

For each of the following, (a) express the probability in words, (b) determine the probability. For conditional probabilities, compute the probabilities two different ways.

1. $p(L)$
2. $p(\overline{L})$
3. $p(G)$
4. $p(\overline{G})$
5. $p(L \cap G)$
6. $p(L \mid G)$
7. $p(G \mid L)$
8. $p(G \mid \overline{L})$
9. $p(\overline{G} \mid L)$

6.2 Product Rule

Applying a little algebra, we get the following product rule:

$$p(A \cap B) = p(A) \cdot p(B \mid A)$$

6.3 Independence

If $p(A) \neq 0$ and $p(B) \neq 0$ and $p(B \mid A) = p(B)$, then we say that A and B are **independent events**.

6.4 Exercises

10. Show that if $p(B \mid A) = p(B)$, then $p(A \mid B) = p(A)$.

11. The product rule is even simpler when A and B are independent. What is it?

12. Show that $p(A \cap B \cap C) = p(A) \cdot p(B \mid A) \cdot p(C \mid A \cap B)$.

13. Analogous rules hold for intersections of more events as well. Write down the rule for the intersection of 4 events.

14. What is the probability of rolling doubles (two numbers that match) with standard dice? Do this two ways: (a) using the Equally Likely Rule and (b) using the Product Rule.

15. What is the probability of a five-card flush (all cards the same suit)? Do this two ways: (a) using the Equally Likely Rule and (b) using the Product Rule.

16. How many people are in this room? What is the probability that two people in this room have the same birthday (month and day)? (Hint: Use the Complement Rule.)

    What assumption must we make to do this calculation?

17. If two 6-sided dice are rolled and the first one is a 5, what is the probability that the sum is 10?

18. If two 6-sided dice are rolled and at least one of them is a 5, what is the probability that the sum is 10?

19. If two 6-sided dice are rolled and the sum is 10, what is the probability that at least one of them is a 5?

7 Probability Trees and Bayes

7.1 Random Primality Testing

Miller's test for primality of a number n uses arithmetic mod n and a base b with $1 < b < n$. For a given b, the test either determines that n is composite or it is inconclusive. It can be shown that for any composite number, the test is conclusive and correct for at least $3/4$ of the possible values of b. For a prime number, the test is always inconclusive. This leads to an interesting randomized test:

```
string PrimeTest(n, attempts) {
  for (i = 0; i < attempts; i++) {
    select b at random with 1 < b < n
    if (MillersTest(n, b) ==  "composite") {
      return("composite (for sure)")
    }
    return("prime (well, probably)")
  }
}
```

a. If n is prime, what is the probability that `PrimeTest(n, 5)` returns `"prime (well, probably)"`?

b. If n is composite, what is the probability that `PrimeTest(n, 5)` returns `"prime (well, probably)"`?

c. If n is composite, what is the probability that `PrimeTest(n, 10)` returns `"prime (well, probably)"`?

d. If n is composite, what is the probability that `PrimeTest(n, 20)` returns `"prime (well, probably)"`?

Note: For RSA purposes, once we get two large probably prime numbers, we can try encoding and decoding a message. If a composite number sneaks through the randomized test, decoding won't work and we can try again with a different pair of probably prime numbers.

7.2 Bob's Beautiful Boxes

Bob has two beautiful boxes of balls. Box A contains 2 green balls and 7 red balls. Box B contains 4 green balls and 3 red balls. Bob flips a coin to randomly select a box. Then he randomly selects one ball from that box. If the selected ball is red, what is the probability that it was chosen from Box A?

Let's build up to this step by step.

a. Is every ball equally likely to be seleced? If not, which are more likely and which less likely?

b. Let's define some events.

- A: Bob selects Box A
- B: Bob selects Box B
- R: Bob selects a red ball

c. Let's do some inventory. Which of these probabilities can we easily determine? Which is our main question? How can you use the items on the list that you know to figure out the values you don't know? (Feel free to add additional probabilities to the inventory if that is helpful.)

$$p(A) \qquad p(B) \qquad p(R)$$

$$p(A \cap B) \qquad p(A \cup B) \qquad p(A \cap R) \qquad p(B \cap R)$$

$$p(A \mid R) \qquad p(R \mid A) \qquad p(B \mid R) \qquad p(R \mid B) \qquad p(R \mid A)$$

7.3 Breast cancer screening

Here is some information about breast cancer screening. (Note: These percentages are approximate, and very difficult to estimate.)

- American Cancer Society estimates that about 1.7% of women have breast cancer. http://www.cancer.org/cancer/cancerbasics/cancer-prevalence
- Susan G. Komen For The Cure Foundation states that mammography correctly identifies about 78% of women who truly have breast cancer. http://ww5.komen.org/BreastCancer/AccuracyofMammograms.html
- An article published in 2003 suggests that up to 10% of all mammograms are false positive. http://www.ncbi.nlm.nih.gov/pmc/articles/PMC1360940

If a mammogram yields a positive result, what is the probability that patient has cancer?

7.4 Disease Testing

A test for a rare medical condition that affects 1 person in 10,000 has the following properties: If a person is healthy, it correctly diagnoses this 98% of the time. If a person is diseased, it correctly diagnoses this 99% of the time. If you take the test and it comes back positive (ie, the test says you have the disease), what is the probability that you have the disease?

Hint: Use the inventory method. Useful events include

- H: person is healthy
- D: person is diseased
- P: test is positive (indicates disease)
- N: test is negative (indicates healthy)

7.5 Hashing

A hash function h assigns one of m storage locations (an index in an array, for example) to each key k. Typically there are many more keys than storage locations, but only a much smaller number of keys will occur in a particular application. For example, if the keys are social security numbers, then there are approximately 350 active social security numbers (keys), but a give application may have data on only a very small fraction of these people. If $h(k_1) = h(k_2)$, then we say there is a collision (because we would like to put both keys into the same storage location and have to figure out some work-around).

Good hash functions have the property that if a key is selected at random, then the probability of being mapped to a specified storage location is approximately $1/m$, where m is the number of storage locations. (So each storage location is roughly equally likely to be selected.) For the questions below, assume that the probability is exactly $1/m$. That should be a good approximation for good hash functions.

a. If a hash function is mapping to 20 storage locations, what is the probability that we can place 4 keys without any colisions? (That is, what is the probability that the first four hash values will all be different.)

b. If a hash function is mapping to 50 storage locations, what is the probability that we can place 10 keys without any colisions? (That is, what is the probability that the first ten hash values will all be different.)

c. If a hash function is mapping to 300 storage locations, what is the probability that we can place 25 keys without any colisions?

d. Make up a similar problem of your own with a different number of storage locations and a different number of keys to place.

e. This is probably too big for your calculator, but if m is 1 million, then the probability of placing 1178 keys without any collisions is $\approx 1/2$.

f. Explain why these results indicate that hash tables will need to deal with colisions if they are going to be memory efficient.

8 Random Variables

8.1 Describing Random Variables

A **random variable** is a random process that produces a number.

- Typically, capital letters (often near the end of the alphabet) are used for random variables. Example:

$$p(X = 3) = 0.2$$

- We will focus our attention on discrete random variables. A discrete random variable can only take on values from a finite or countable list of values. We will be especially interested in the finite case.
- Any random process can be converted into a random variable by a function that maps outcomes in the sample space to numbers. In our text you will see notation like $X : S \to R$ (X maps each outcome to a number) or $X(HHT) = 2$ (counting the number of heads).
- We will mostly work with random variables in two formats: (a) a table of values and probabilities, or (b) a formula that tells how to compute the probability for any specified value.

8.2 Probability Tables

First example

Here is an example probability table for a random variable we'll call X.

value	1	2	3	4
probability	0.4	0.3	0.2	0.1

1. What properties must the numbers in such a table always have?
2. What is $p(X = 2)$?
3. What is $p(X > 2)$?
4. What is $p(X \text{ is even})$?

Creating a probability table

5. Create a probability table for the random variable H, where H is the number of heads observed when flipping 4 fair coins. Start by determining all the possible values. Then for each value, determine its probability.
6. Let X be the absolute value of the difference in the values of two fair 4-sided dice. For example, if you roll a 3 and a 1, the difference is 2. Create a probability table for X. (What are the possible values?)
7. Let D be the number of diamonds dealt in a 5 card hand.
   a. Create a probability table for D.
   b. What is the most likely number of diamonds in such a hand?

8.3 Probability Functions

We can also describe random variables with formulas.

8. Let Y be defined by $p(Y = y) = y/10$ when $y \in \{1, 2, 3, 4\}$.

   a. What is $p(Y = 3)$?

   b. Create a probability table for Y.

9. Suppose you keep flipping a coin until you get a head. Let F count the number of flips.

   a. What is $p(F = 1)$?

   b. What is $p(F = 2)$?

   c. What is $p(F = 3)$?

   d. What are the possible values for F?

   e. Can you come up with a formula for $p(F = k)$?

10. Let X be the number of heads in n tosses of a fair coin.

    a. What are the possible values of X?

    b. Give a formula for $p(X = k)$?

    c. Give a formula for $p(X = x)$? (This is the same as the preceding part, but with different notation.)

11. Now suppose the coin isn't fair. Let p be the probability of getting a head on each flip. Let X be the number of heads in n flips.

    a. What is $p(X = 1)$ when $n = 1$ and $p = 0.75$?

    b. What is $p(X = 1)$ when $n = 2$ and $p = 0.75$?

    c. Make a probability table for X when $n = 3$ and $p = 0.75$.

    d. Determine a general formula for $p(X = x)$ in terms of n and p.

8.4 Binomial Distributions

The previous example illustrates a scenario that arises in many applications:

1. A random process consists for n parts or steps (traditionally called trials).
2. Each trial has two possible outcomes (traditionally called success and failure).
3. The probability of success is the same (p) for each trial.
4. The trials are independent of each other.
5. Our random variable is the count of the number of successes in the n trials.

We will denote this $X \sim \mathsf{Binom}(n, p)$.

11. Create a probability table for a $\mathsf{Binom}(2, 0.2)$ random variable.

12. Create a probability table for a $\mathsf{Binom}(3, 1/4)$ random variable.

9 Expected Value

9.1 An Example: GPA

Question: If a student receives 5 A's, 4 B's, and 1 C, what is the student's GPA? (Assume all courses are equally weighted and that an A is worth 4, a B with 3, and a C worth 2.)

1. Explain why $\frac{4+3+2}{3}$ is *not* the correct answer.
2. Show how to correctly calculate the GPA by listing the 10 scores individually.
3. Factor your expression into the following form (fill in the missing numerators):

$$\text{GPA} = 4 \cdot \frac{\quad}{10} + 3 \cdot \frac{\quad}{10} + 2 \cdot \frac{\quad}{10}$$

4. Let X be the random variable that results from randomly selecting a course and recording its grade (on a 4 point scale). Create a probability table for X.
5. How are the numbers in your expression for GPA in #3 related to the numbers in your probability table?

9.2 Generalizing

We can generalize this idea to compute the **mean** (more commonly called **expected value**) of any random variable X. This is denoted either $\mathrm{E}(X)$ or μ_X.

6. Let X be defined by the probability table below. Compute $\mathrm{E}(X)$.

value of X	0	1	2	3
probability	0.2	0.3	0.4	0.1

7. Let H be the number of heads in 3 tosses of a fair coin.
   a. Create a probability table for H
   b. Compute $E(H)$.
8. Use good mathematical notation to write down the definition:

$$\mathrm{E}(X) =$$

9. A raffle has 1000 tickets. Holders of 4 of the tickets get a prize. The other 996 are worth nothing. The four prizes are worth \$500, \$200, \$50, and \$50. Let V be the value of a random raffle ticket.
   a. Create a probabilty table for V.
   b. Compute $\mathrm{E}(V)$.
   c. What does $\mathrm{E}(V)$ tell us about the raffle tickets?

10. Let D be the absolute value of the difference between the values of **two 4-sided dice**.
   a. Create a probability table for D.
   b. Compute $\mathrm{E}(D)$.
   c. What is $p(D = \mathrm{E}(D))$?
   d. What is $p(D < \mathrm{E}(D))$?
   e. What is $p(D > \mathrm{E}(D))$?

We can do these already now, but there are easier ways that take advantage of properties of expected value that we haven't learned yet.

11. In a hand of 5 cards from a standard deck, what is the expected number of diamonds?

12. If you roll 5 standard dice, what is the expected number of 6's?

These two are a bit more challenging, but still doable.

13. In a hand of 5 cards from a standard deck, what is the expected number of suits?

14. If you roll 5 standard dice, what is the expected number of unique numbers rolled?