Random Variables

Stat 241

Discrete Random Variable

A random variable X is a **discrete random variable** if the range of X is finite or an infinite sequence of values. A discrete random variable can be characterized by a function called a **probability mass function (pmf)** which specifies the probability for each possible numerical value of the random variable.

$$p(x) = \mathrm{P}(X = x)$$

This function can be specified with a formula or in a table of probabilities. Given the pmf we can calculate the probability any event by adding the probabilities for all of the these values in the event.

Example 1. Toss a fair coin 5 times. Let X = number of heads produced.

a. Create a probability table for X.
b. What is $\mathrm{P}(X \text{ is even})$?
c. What is $\mathrm{P}(X \geq 2)$?

Binomial Random Variables

This example is an instance of an important kind of discrete random variabl called a **binomial random variable**. Binomial random variables arise in situations which have the following properties.

1. The random process consists of a **predetermined number of trials** (usually denoted n).
2. Each trial has **two outcomes** (generically called success and failure).
3. The **probability** of success **is the same for each trial** (often denoted p).
4. Each trial is **independent** of the others.
5. The random variable counts the **number of successes** in the n trials.

Such a variable is dennoted by $X \sim \mathsf{Binom}(n, p)$.

There is a handy formula for the pmf of a $\mathsf{Binom}(n, p)$ random variable:

$$p(x) = \mathrm{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$. $\binom{n}{x}$ is read "n choose x" and counts the number of ways to pick a set of x items from a collection of size n. This can be computed in R using the `choose()` function.

```r
choose(5, 2)
```

```
## [1] 10
```

```r
factorial(5) / (factorial(2) * factorial(3))
```

```
## [1] 10
```

Example 2. If we roll a fair die 10 times, what is the probability of getting 2 or more sixes?

Example 3. Two cards are dealt from an ordinary deck of playing cards. Let X = number of aces dealt.

a. Create a probability table for X.
b. Is X a binomial random variable? If so, what are n and p? If not, why not?
c. What is $P(X \geq 1)$?

Example 4. A fair coin is tossed until a head is produced. Let X = number of tosses.

a. Create a probability table for X.
b. Can you give a formula for the pmf for X?
c. Can you show that the sum of all the probabilities is 1?
d. What is $P(X \geq 4)$?
e. What is the probability it takes an even number of tosses to production first head? ($P(X$ is even)

Continuous Random Variables

A **continuous random variable** is a random variable whose range is an interval of real numbers.

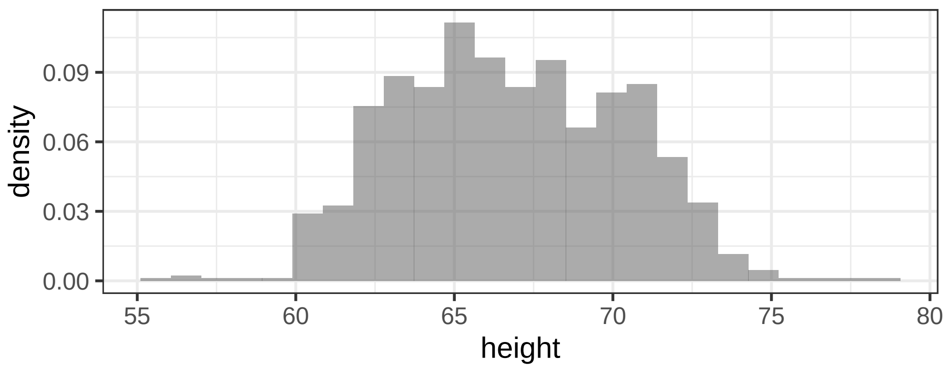
Examples

- Randomly select an adult male: Let X = (exact) height in inches
- Randomly select a washer produced by a particular machine. Let X = (exact) diameter in mm.
- Randomly select a battery and use it to power a device. Let X = (exact) time until battery can no longer power device.

Density histograms and density plots of data

Let's consider the heights of the adult children in the `Galton` data set. Here is a **density histogram**.

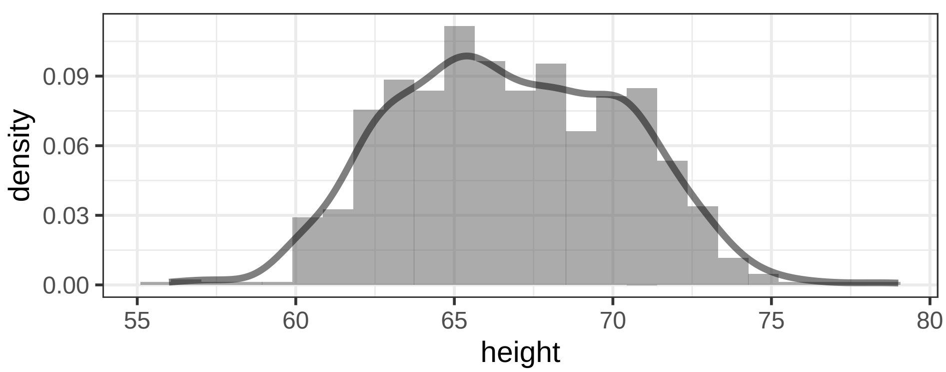
```
gf_dhistogram( ~ height, data = Galton)
```



The density scale is chosen so that ____.

Now let's look at a density plot.

```
gf_dhistogram( ~ height, data = Galton) %>%
gf_dens( ~ height, data = Galton, size = 1.2)
```



This provides a "smooth" version of the histogram and also has the property that

Density Curves and Density Functions

A continuous random variable is described by a **probability density function (pdf)**. The plot of a pdf will look just like curve in a density plot.

Probability density functions always have two important properties:

1.

2.

We determine probabilities from a pdf but taking the area under the curve over the region corresponding to our event (ie, by integration).

$$\mathrm{P}(a \leq X \leq b) = \int_a^b f(x)\, dx$$

Example 5. Let X be a number randomly chosen from the interval $[0, 2]$ in such a way that all numbers are equally likely. We call $[0, 2]$ the **support** of X is [0,2].
Since no value of X is more likely to be selected than any other value, the density function $f(x)$ must be a constant on [0,2].

a. What is the constant value?

b. What is $\mathrm{P}(X = 1)$?

c. What is $\mathrm{P}(0 \leq X \leq 1)$?

d. Waht is $\mathrm{P}(1 \leq X \leq 3/2)$?

e. What is $\mathrm{P}(1 \leq X \leq 3)$?

A random variable X whose pdf is constant (where it is non-zero) is said to have a **uniform distribution**. We will denote this as $X \sim \mathsf{Unif}(a, b)$, where a and b are the upper and lower limits of the support.

Example 6. Let $f(x) = x/2$ for $x \in (0, 2)$ (and 0 elsewhere). We can write this as $f(x) = x/2 \cdot [\![x \in (0, 2)]\!]$ or $f(x) = x/2$ on $(0, 2)$.

a. Verify that $f(x)$ is a probability density function.

b. Compute $\mathrm{P}(1 \leq X \leq 3/2)$.

c. Compute $\mathrm{P}(1 \leq X \leq 4)$.

The Cumulative Distribution Function (cdf) for a continuous random variable

The **cdf** for X is defined by

$$F(x) = \mathrm{P}(X \leq x)$$

Example 7. Let X have a uniform distribution on $[0, 4]$.

a. pdf: $f(x) =$

b. cdf: $F(x) =$

Example 8. Let X be the random variable whose pdf is $f(x) = x/2 \cdot [\![x \in (0, 2)]\!]$. Find the cdf for X.

Using the cdf to compute probabilities

If $F(x)$ is the cdf for the random variable X, then $\mathrm{P}(a \leq X \leq b) =$ ______________________________

Example 9. Let $F(x) = x^2$ on $[0, 1]$

a. What is $F(x)$ when $x < 0$?

b. What is $F(x)$ when $x > 0$?

c. What is $P(X \leq 1/2)$?

d. What is $P(1/2 \leq X \leq 3/4)$?

e. What is $P(-2 \leq X \leq 1/2)$?

f. What is the pdf for X?

Example 10. Let $f(x) = e^{-x}$ on $(0, \infty)$ (and 0 elsewhere).

a. Show that f is a pdf.
b. Let X be the corresponding random variable. What is $\mathrm{P}(X > 2)$?
c. What is the cdf for X?
d. Use the cdf to find $\mathrm{P}(X < 1)$.

More Practice

1. Parts coming off an assembly line have a 1% chance of being defective. If 3 parts are randomly chosen from this line and X is the number of defective parts.

a. Compute the probability function p(x) for X.

b. What is the probability that at least one of the three is defective?

2. Parts coming off an assembly line have a 1% chance of being defective. All of the parts coming off the line are inspected. Let X be the number parts inspected up to and including the first defective part.

a. Is X continuous or discrete?

b. What is the support of X?

c. Find the probability mass function $p(x)$ for X.

d. What is the probability that the first defective part is the 100th part?

3. A biased coin has a 40% chance of producing a head. If it is tossed 10 times,

a. What is the probability of getting exactly 3 heads?

b. What is the probability of getting 3 or more heads. (This can be calculated in two different ways. The easier way uses the complement rule.)

4. a. Find the value of C for which the function $f(x) = Cx^2 \cdot [\![x \in (0, 2)]\!]$ is a pdf.

b. Use the pdf to find $\text{P}(0 \leq X \leq 1)$ and $\text{P}(1 \leq X \leq 5)$.

c. Use the pdf to compute the cumulative distribution function $F(x)$.

5. a. Find the value of C for which the function $f(x) = \frac{C}{x^2} \cdot [\![x \geq 1]\!]$ is a pdf.

b. What is $\text{P}(X \leq 2)$?

c. What is $\text{P}(X > 3)$?

d. Find the cumulative distribution function $F(x)$.

e. Use the cumulative distribution function to find $\text{P}(2 \leq X \leq 5)$.