Conditional Probability

Stat 241

$\mathrm{P}(A \mid B)$ answers the question: Of the times that B happens, how often does A also happen? Common ways this is expressed include

- The probability of A given B
- The probability of A conditional on B
- The probability of A if B
- The probability that A happens when B happens

When $\mathrm{P}(B) \neq 0$, then

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \,\mathrm{and}\, B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(\mathrm{both})}{\mathrm{P}(\mathrm{condition})}$$

Note: Usually, $\mathrm{P}(A \mid B)$, $\mathrm{P}(B \mid A)$ and $\mathrm{P}(A \,\mathrm{and}\, B)$ are all different. It is critical to know which of these three applies in a given situation and to use the notation correctly.

Practice with Conditional Probability

A group of 10 men and 15 women were polled as to whether they preferred eating an orange or eating an apple. The results of the poll are given below

	apple	orange
female	10	5
male	4	6

Suppose we put the names of all the people into a hat and select one randomly. Consider the following events.

- M: The selected person is Male.
- A: The selected person's prefers apples to oranges.

For each of the following, (a) express the probability in words, (b) determine the probability. For conditional probabilities, compute the probabilities two different ways.

1. $\mathrm{P}(A)$
2. $\mathrm{P}(\mathrm{not}\, A)$
3. $\mathrm{P}(M)$
4. $\mathrm{P}(\mathrm{not}\, M)$
5. $\mathrm{P}(A \,\mathrm{and}\, M)$
6. $\mathrm{P}(A \mid M)$
7. $\mathrm{P}(M \mid A)$
8. $\mathrm{P}(M \mid \mathrm{not}\, A)$
9. $\mathrm{P}(\mathrm{not}\, M \mid A)$

Product Rule

Applying a little algebra, we get the following product rule:

$$\mathrm{P}(A \,\mathrm{and}\, B) = \mathrm{P}(A) \cdot \mathrm{P}(B \mid A)$$

Independence

If $\mathrm{P}(A) \neq 0$ and $\mathrm{P}(B) \neq 0$ and $\mathrm{P}(B \mid A) = \mathrm{P}(B)$, then we say that A and B are **independent events**.

Intuitively, A and B are independent when the probability that one of them occurs does not depend on whether the other one occurred.

More Practice

1. The product rule is even simpler when A and B are independent. What is it?
2. Show that $\mathrm{P}(A \,\mathrm{and}\, B \,\mathrm{and}\, C) = \mathrm{P}(A) \cdot \mathrm{P}(B \mid A) \cdot \mathrm{P}(C \mid A \,\mathrm{and}\, B)$.
3. Analogous rules hold for intersections of more events as well. Write down the rule for the intersection of 4 events.
4. What is the probability of rolling doubles (two numbers that match) with standard dice? Do this two ways: (a) using the Equally Likely Rule and (b) using the Product Rule.
5. What is the probability of a five-card flush (all cards the same suit)?
6. There are 18 students in our class. What is the probability that two people in this class have the same birthday (month and day)? (Hint: Use the Complement Rule.)

   What assumption must we make to do this calculation? Is that a reasonable assumption?
7. If two 6-sided dice are rolled and the first one is a 5, what is the probability that the sum is 10?
8. If two 6-sided dice are rolled and at least one of them is a 5, what is the probability that the sum is 10?
9. If two 6-sided dice are rolled and the sum is 10, what is the probability that at least one of them is a 5?
10. A pair of fair dice is tossed. Given that the sum of the dice is 7, what is the probability that one of the dice came up 3?
11. A jar contains 5 red and 4 green chips. Two chips are drawn without replacement . What is the probability that both are red? What is the probability that one is red and one is green?
12. If in the previous example, the drawing is with replacement, what are the probabilities of the events?
13. Four cards are dealt from an ordinary deck of playing cards. What is the probability that they are all aces?
14. You have two jars of chips. Jar 1 contains 4 red and 4 blue chips. Jar 2 contains 5 red and 3 blue chips. You toss a fair coin. If the coin toss is heads, you draw a chip from Jar 1. If it is tails, you draw from Jar 2.
    What is the probability the you draw a red chip? What is the probability that if the drawn chip is red, it came from Jar1 (the coin was heads)?
15. A new test has been developed for a serious disease. Like all such tests it's not perfect. It has a false positive rate of 2%; i.e., if a person doesn't have the disease, there is a 2% chance that the test will be positive. It also has a false negative rate of 3%; i.e., if a person has the disease, there is a 3% chance that the test is negative. It is a relatively uncommon disease, affecting only 0.1% of the population. How good is the test? If you test positive, what's the probability that you have the disease?
16. A fair die is tossed 3 times. What is the probability that a 1 occurs on each toss?

17. A manufacturer claims that 99% of its product will still be functioning after 3 years. I buy 5 of the company's product.

    a. What is the probability that all 5 of these items will still be functioning after 5 years?
    b. What is the probability that exactly 4 of these items will still be functioning after 5 years?
    c. What assumption must we make about these products for these probability calculations to be correct?

18. Show that if $\mathrm{P}(B \mid A) = \mathrm{P}(B)$, then $\mathrm{P}(A \mid B) = \mathrm{P}(A)$.